New global stability estimates for the Gel'fand-Calderon inverse problem

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Abstract. We prove new global stability estimates for the Gel'fand-Calderon inverse problem in 3D. For sufficiently regular potentials this result of the present work is a principal improvement of the result of [G.Alessandrini, Stable determination of conductivity by boundary measurements, Appl. Anal. **27** (1988), 153-172].

1. Introduction

We consider the equation

$$-\Delta\psi + v(x)\psi = 0, \quad x \in D, \tag{1.1}$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d, \ \ d \geq 2, \ \partial D \in C^2, \ v \in L^\infty(D). \tag{1.2}$$

Equation (1.1) arises, in particular, in quantum mechanics, acoustics, electrodynamics. Formally, (1.1) looks as the Schrödinger equation with potential v at zero energy.

We consider the map Φ such that

$$\frac{\partial\psi}{\partial\nu}\big|_{\partial D} = \Phi(\psi\big|_{\partial D}) \tag{1.3}$$

for all sufficiently regular solutions ψ of (1.1) in $\bar{D} = D \cup \partial D$, where ν is the outward normal to ∂D. Here we assume also that

$$0 \text{ is not a Dirichlet eigenvalue for the operator } -\Delta + v \text{ in } D. \tag{1.4}$$

The map Φ is called the Dirichlet-to-Neumann map for equation (1.1) and is considered as boundary measurements for (physical model described by) (1.1).

We consider the following inverse boundary value problem for equation (1.1):

Problem 1.1. Given Φ, find v.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at zero energy (see [9], [16]). This problem can be also considered as a generalization of the Calderon problem of the electrical impedance tomography (see [5], [23], [16]).

We recall that the simplest interpretation of D, v and Φ in the framework of the electrical impedance tomography consists in the following (see, for example, [16], [14], [13]): D is a body with isotropic conductivity $\sigma(x)$ (where $\sigma \geq \sigma_{min} > 0$),

$$v(x) = (\sigma(x))^{-1/2}\Delta(\sigma(x))^{1/2}, \quad x \in D, \tag{1.5}$$

$$\Phi = \sigma^{-1/2}\big(\Lambda\sigma^{-1/2} + \frac{\partial\sigma^{1/2}}{\partial\nu}\big), \tag{1.6}$$

where Δ is the Laplacian, Λ is the voltage-to-current map on ∂D, and $\sigma^{-1/2}$, $\partial\sigma^{1/2}/\partial\nu$ in (1.6) denote the multiplication operators by the functions $\sigma^{-1/2}\big|_{\partial D}$, $(\partial\sigma^{1/2}/\partial\nu)\big|_{\partial D}$, respectively. In addition, (1.4) is always fulfilled if v is given by (1.5) (where $\sigma \geq \sigma_{min} > 0$ and σ is twice differential in $L^\infty(D)$).

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness theorems for Problem 1.1 (in its Calderon or Gel'fand form) in dimension $d \geq 3$ were obtained for the first time in [23] and [16]. In particular, according to [16], under assumptions (1.2), (1.4), the map Φ uniquely determines v.

A global reconstruction method for Problem 1.1 in dimension $d \geq 3$ was proposed for the first time in [16]. In its simplest form, this method of [16] consists in reducing Problem 1.1 to Problem 4.1 via formulas and equations (5.1), (5.2) for $v_2 = v$, $\psi_2 = \psi$, $h_1 = h$, $v_1 \equiv 0$, $\psi_1(x,k) = e^{ikx}$, $R_1(x,y,k) = G(x-y,k)$, $h_1 \equiv 0$ and in solving Problem 4.1 via formula (4.10), see Sections 3,4 and 5 of the present paper.

Global stability estimates for Problem 1.1 in dimension $d \geq 3$ were given for the first time in [1]. A variation of this result is presented as Theorem 2.1 of Section 2 of the present paper.

We recall that in global results one does not assume that σ or v is small in some sense or that σ or v is peicewise constant or piecewise real-analytic. For peicewise constant or piecewise real-analytic σ the first uniqueness results for the Calderon version of Problem 1.1 in dimension $d \geq 2$ were given in [6], [11].

As regards global results given in the literature on Problem 1.1 in dimension $d = 2$, the reader is refered to Corollary 2 of [16] and to [15], [3], [13], [12], [4] and to references given in [4].

In the present work we continue studies of [16], [17], [19], [21], [20]. One of the main purposes of these studies was developing an efficient reconstruction algorithm for Problem 1.1 in dimension $d = 3$. Note that the aforementioned global reconstruction method of [16] is fine in the sense that it consists in solving Fredholm linear integral equations of the second type and using explicit formulas but this reconstruction is not optimal with respect to its stability properties. An effectivization of this reconstruction of [16] with respect to its stability properties was developed in [17], [19], [20]. In [21] and in the present work we illustrate our progress in stability by proving new stability estimates for Problem 1.1 in 3D and, in particular, by proving Theorem 2.2 of Section 2 in 3D. For sufficiently regular potentials this theorem is a principle improvement of the aforementioned Alessandrini stability result of [1].

Note that algorithms developed in [17], [19], [20] for finding v from Φ in 3D can be used even if v has discontinuities. In this case these algorithms can be considered as methods for finding the smooth part of v from Φ. In particular, the global algorithms of [17], [20] can be considered as methods for approximate finding the Fourier transform $\hat{v}(p)$, $|p| \leq 2\rho$, from Φ for sufficiently great ρ, where $\hat{v}(p)$ is defined by (4.12), $d = 3$. However, for the case when v is not smooth enough we did not manage yet to formalize our principle progress of [20] in global stability as a rigorous mathematical theorem.

Note that in [21] our new stability estimates of Theorem 2.2 of the next section were proved in the Born approximation (that is in the linear approximation near zero potential) only. Besides, a scheme of proof of these estimates was also mentioned in [21] for potentials with sufficiently small norm in dimension $d = 3$. In the present work we give a complete proof of Theorem 2.2 in the general (or by other words global) case in dimension $d = 3$.

Our new stabilty estimates for Problem 1.1 (that is the estimates of Theorem 2.2) are presented and discussed in Section 2. In addition, relations between Theorems 2.1, 2.2 and global reconstructions of [16], [17], [20] are explained in Section 3. The proof of Theorem 2.2 for the global case in 3D is given in Section 8.

2. Stability estimates

As in [21] we assume for simplicity that

$$\begin{aligned} &D \ \text{ is an open bounded domain in } \ \mathbb{R}^d, \ \partial D \in C^2, \\ &v \in W^{m,1}(\mathbb{R}^d) \ \text{ for some } m > d, \ supp\, v \subset D, \ d \geq 2, \end{aligned} \tag{2.1}$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v : \ \partial^J v \in L^1(\mathbb{R}^d), \ |J| \leq m\}, \quad m \in \mathbb{N} \cup 0, \tag{2.2}$$

where

$$J \in (\mathbb{N} \cup 0)^d, \quad |J| = \sum_{i=1}^{d} J_i, \quad \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}.$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \|\partial^J v\|_{L^1(\mathbb{R}^d)}. \tag{2.3}$$

Let

$$\begin{aligned} &\|A\| \ \text{ denote the norm of an operator} \\ &A : \ L^\infty(\partial D) \to L^\infty(\partial D). \end{aligned} \tag{2.4}$$

We recall that if v_1, v_2 are potentials satisfying (1.2), (1.3), where D is fixed, then

$$\Phi_1 - \Phi_2 \ \text{ is a compact operator in } \ L^\infty(\partial D), \tag{2.5}$$

where Φ_1, Φ_2 are the DtN maps for v_1, v_2 respectively, see [16], [17]. Note also that (2.1) $\Rightarrow$ (1.2).

Theorem 2.1 (variation of the result of [1]). *Let conditions* (1.4), (2.1) *hold for potentials* v_1 *and* v_2, *where* D *is fixed,* $d \geq 3$. *Let* $\|v_j\|_{m,1} \leq N$, $j = 1, 2$, *for some* $N > 0$. *Let* Φ_1, Φ_2 *denote the DtN maps for* v_1, v_2, *respectively. Then*

$$\|v_1 - v_2\|_{L^\infty(D)} \leq c_1 (\ln(3 + \|\Phi_1 - \Phi_2\|^{-1}))^{-\alpha_1}, \tag{2.6}$$

where $c_1 = c_1(N, D, m)$, $\alpha_1 = (m - d)/m$, $\|\Phi_1 - \Phi_2\|$ *is defined according to* (2.4).

As it was mentioned in [21], Theorem 2.1 follows from formulas (4.9)-(4.11), (5.1) (of Sections 4 and 5).

A disadvantage of estimate (2.6) is that

$$\alpha_1 < 1 \ \text{ for any } \ m > d \ \text{ even if } \ m \ \text{ is very great.} \tag{2.7}$$

Theorem 2.2. *Let the assumptions of Theorem 2.1 hold. Then*

$$\|v_1 - v_2\|_{L^\infty(D)} \le c_2(\ln(3 + \|\Phi_1 - \Phi_2\|^{-1}))^{-\alpha_2}, \tag{2.8}$$

where $c_2 = c_2(N, D, m)$, $\alpha_2 = m - d$, $\|\Phi_1 - \Phi_2\|$ *is defined according to* (2.4).

A principal advantage of estimate (2.8) in comparison with (2.6) is that

$$\alpha_2 \to +\infty \quad \text{as} \quad m \to +\infty, \tag{2.9}$$

in contrast with (2.7).

In the Born approximation, that is in the linear approximation near zero potential, Theorem 2.2 was proved in [21].

For sufficiently small N in dimension $d = 3$, a scheme of proof of Theorem 2.2 was also mentioned in [21]. This scheme involves, in particular, results of [17], [19].

In the general (or by other words global) case Theorem 2.2 in dimension $d = 3$ is proved in Section 8. This proof involves, in particular, results of [17], [20]. We see no principal difficulties (except restrictions in time) to give a similar proof in dimension $d > 3$.

We would like to mention that, under the assumptions of Theorems 2.1 and 2.2, according to the Mandache results of [13], the estimate (2.8) can not hold with $\alpha_2 > m(2d - 1)/d$. However, we think that this instability result of [13] can be sharpened considerably and that in this sense our Theorem 2.2 is almost optimal already.

For additional information concerning stability and instability results given in the literature for Problem 1.1 (in its Calderon or Gel'fand form) the reader is refered to [1], [13], [21], [22], [24] and references therein. In particular, in [24] estimate (2.6) is proved for $d = 2$ and any $\alpha_1 \in]0, 1[$, under the assumptions that $v_j \in C^2(\bar D)$, $\|v_j\|_{C^2(\bar D)} \le N$ (and, at least, under the additional assumption that $supp\, v_j \subseteq \bar D_0 \subset D$ for some closed $\bar D_0$), $j = 1, 2$, where c_1 depends on D, N, α_1 (and on $\bar D_0$).

3. Theorems 2.1 and 2.2 and global reconstructions of [16], [17], [20]

In this section we explain relations between Theorems 2.1, 2.2 and global reconstructions of [16], [17], [20] for Problem 1.1, where we assume that dimension $d = 3$. (The case $d > 3$ is similar to the case $d = 3$.) The scheme of the aforementioned global reconstructions consists in the following:

$$\Phi \to \ h\big|_\Xi \to \hat v_{\rho,\tau} \to v, \tag{3.1}$$

where h is the Faddeev generalized scattering amplitude (at zero energy) defined in the complex domain Θ (see formulas (4.4), (4.5), (4.1)-(4.3) of Section 4 for $d = 3$), Ξ is an appropriate subset of Θ, for example Ξ is the subset of Θ with the imaginary part length equal to ρ, $\hat v_{\rho,\tau}$ is an approximation to $\hat v$ on $\mathcal{B}_{2\tau\rho}$,

$$\hat v_{\rho,\tau} \to \hat v \ \text{ on } \ \mathcal{B}_{2\tau\rho} \ \text{ as } \ \rho \to +\infty, \tag{3.2}$$

where $\hat v$ is the Fourier transform of v (see formula (4.12), $d = 3$), $\mathcal{B}_r$ is the ball of the radius r, τ is a fixed parameter, $0 < \tau \le 1$.

3.1. Theorem 2.1 and global reconstructions of [16], [17]. For the case when

$$\Xi = b\Theta_\rho \quad \text{(see formulas (6.2))}, \tag{3.3a}$$

$$\begin{aligned}
&h\big|_{b\Theta_\rho} \ \text{ is determined from } \ \Phi \ \text{ via} \\
&(5.1), (5.2) \ \text{ considered for } \ v_2 = v,\ \psi_2 = \psi,\ h_2 = h, \\
&v_1 \equiv 0,\ \psi_1(x,k) = e^{ikx},\ R_1(x,y,k) = G(x-y,k),\ h_1 \equiv 0,
\end{aligned} \tag{3.3b}$$

$$\begin{aligned}
&\hat{v}_{\rho,\tau}(p) = h(k(p), l(p)),\ p \in \mathcal{B}_{2\tau\rho},\ \tau = 1, \\
&\text{for some vector} - \text{functions} \ \ k(p),\ l(p) \\
&\text{such that} \ \ k(p), l(p) \in b\Theta_\rho, \quad k(p) - l(p) = p,
\end{aligned} \tag{3.3c}$$

reconstruction (3.1) was, actually, given in [16]. In this case (3.2) holds according to (4.10), (4.11). (In the framework of [16] less precise version of (4.11) was known.)

General formulas and equations (5.1)-(5.3) for finding $h\big|_{b\Theta_\rho}$ from Φ with known background potential $v_1 \not\equiv 0$ were given for the first time in [17]. If unknown potential v is sufficiently close to some known background potential $v_1 \not\equiv 0$, then formulas and equations (5.1)-(5.3) considered for $v_2 = v$, $\psi_2 = \psi$, $h_2 = h$ give more stable determination of $h\big|_{b\Theta_\rho}$ from Φ than these formulas and equations for $v_1 \equiv 0$ (see [17] for more detailed discussion of this issue).

Note that Theorem 2.1 follows from formulas (4.10), (4.11), (5.1) mentioned above in connection with the versions of (3.1), (3.2) of [16], [17].

3.2. Theorem 2.2 and global reconstructions of [17], [20]. The main disadvantage of the global reconstruction (3.1), (3.2) in the framework of [16], [17] is related with the following two facts:

(1) The determination of $h\big|_{\Xi}$ for $\Xi = b\Theta_\rho$ or, more generally, for $\Xi \subseteq \bar{\Theta}_\rho$, $\Xi \cap b\Theta_\rho \neq \emptyset$ (see formulas (6.2)) from Φ via formulas and equations (5.1)-(5.3) for $v_2 = v$, $\psi_2 = \psi$, $h_2 = h$ with some known background potential v_1 is stable for relatively small ρ, but is very unstable for $\rho \to +\infty$ in the points of Ξ with sufficiently great imaginary part (see [17], [19], [20] for more detailed discussion of this issue).

(2) For $\hat{v}_{\rho,\tau}$ defined as in (3.3c) the decay of the error $\hat{v} - \hat{v}_{\rho,\tau}$ on $\mathcal{B}_{2\rho\tau}$ is very slow for $\rho \to +\infty$ (not faster than $O(\rho^{-1})$ even for infinitely smooth compactly supported v) (see [20] for more detailed discussion of this issue).

As a corollary, the global reconstruction (3.1), (3.2) in the framework of [16] and even in the framework of [17] is not optimal with respect to its stability properties. A principle effectivization of the global reconstruction (3.1), (3.2) with respect to its stability properties was given in [20].

The main new result of [20] consists in stable construction of some $\hat{v}_{\rho,\tau}$ on $\mathcal{B}_{2\tau\rho}$ from h on $b\Theta_{\rho,\tau} \subseteq b\Theta_\rho$ (see definitions (6.2)) in such a way that

$$\begin{aligned}
&|\hat{v}(p) - \hat{v}_{\rho,\tau}(p)| \leq C(m, \mu_0, \tau) N^2 (1+|p|)^{-\mu_0} \rho^{-(m-\mu_0)} \\
&\text{for} \ \ p \in \mathcal{B}_{2\tau\rho},\ \rho \geq \rho_1 \ \text{ and fixed } \tau \in]0, \tau_1[,
\end{aligned} \tag{3.4}$$

under the assumptions that

$$\begin{aligned}&v \in W^{m,1}(\mathbb{R}^2),\ \ \|v\|_{m,1} < N \ \ \text{(see definitions (2.2), (2.3))},\\ &m > 2,\ 2 \le \mu_0 < m,\ \tau_1 = \tau_1(m, \mu_0, N),\ \ \rho_1 = \rho_1(m, \mu_0, N),\end{aligned} \tag{3.5}$$

where $C(m, \mu_0, \tau)$, $\tau_1(m, \mu_0, N)$, $\rho_1(m, \mu_0, N)$ are special constants and, in particular, $0 < \tau_1 < 1$, see Theorem 2.1 of [20]. The scheme of this construction of [20] consists in the following:

$$\begin{aligned}&h\big|_{b\Theta_{\rho,\tau}} \to H\big|_{b\Omega_{\rho,\tau}} \to H\big|_{b\Lambda_{\rho,\tau,\nu}}\\ &\to H^0\big|_{\Lambda_{\rho,\tau,\nu}} \to \tilde H_{\rho,\tau} \ \text{ on } \Lambda_{\rho,\tau,\nu}\\ &\to \hat v_{\rho,\tau} \ \text{ on } \ \mathcal{B}_{2\tau\rho}.\end{aligned} \tag{3.6}$$

Here $h\big|_{b\Theta_{\rho,\tau}}$ and $H\big|_{b\Omega_{\rho,\tau}}$ are related by (4.13) (see also the definitions of $b\Theta_{\rho,\tau}$ and $b\Omega_{\rho,\tau}$ given in (6.2), (6.3)), $H\big|_{b\Lambda_{\rho,\tau,\nu}}$ is defined in terms of $H\big|_{b\Omega_{\rho,\tau}}$ by means of (7.4) (where $b\Lambda_{\rho,\tau,\nu}$ is defined in (6.13)), $H^0\big|_{\Lambda_{\rho,\tau,\nu}}$ is defined in terms of $H\big|_{b\Lambda_{\rho,\tau,\nu}}$ by (7.8) (see also formulas (6.13), (6.16)), $\tilde H_{\rho,\tau}$ is defined as the solution of (7.7) for H on $\Lambda_{\rho,\tau,\nu}$, where $Q_{\rho,\tau}$ is replaced by zero in (7.7), finally, $\hat v_{\rho,\tau}$ on $\mathcal{B}_{2\tau\rho}$ is defined as $\hat v_{\rho,\tau} = \hat v^+_{\rho,\tau}$ or as $\hat v_{\rho,\tau} = \hat v^-_{\rho,\tau}$, where

$$\begin{aligned}\hat v^+_{\rho,\tau}(p) &= \lim_{\lambda\to 0} \tilde H_{\rho,\tau}(\lambda, p),\\ \hat v^-_{\rho,\tau}(p) &= \lim_{\lambda\to \infty} \tilde H_{\rho,\tau}(\lambda, p),\end{aligned} \tag{3.7}$$

where definitions (3.7) are similar to formulas (7.16). For more detailed discussion of (3.6) see Section 5 of [20].

Note that in [20] we deal with approximate finding v from $h\big|_{b\Theta_\rho}$ in a similar way with approximate finding v from the scattering amplitude f at fixed energy E of [18]. The parameter ρ of [20] plays the role of the parameter $\sqrt{E}$ of [18].

One can see that for sufficiently regular potentials v that is for $v \in W^{m,1}(\mathbb{R}^3)$, where m is sufficiently great, and for $\hat v_{\rho,\tau}$ constructed from $h\big|_{b\Theta_{\rho,\tau}}$ via (3.6) as in [20] the decay rate of the error $\hat v - \hat v_{\rho,\tau}$ on $\mathcal{B}_{2\tau\rho}$ is $O(\rho^{-(m-2)})$ as $\rho \to +\infty$ (see (3.4) for $\mu_0 = 2$) and is much faster than for $\hat v_{\rho,\tau}$ given by (3.3c) as in [16], [17]. Therefore, for Problem 1.1 in dimension $d = 3$ the global reconstruction of [20] (using also (5.1)-(5.3) for finding $h\big|_{b\Theta_{\rho,\tau}}$ from Φ as in [17]) has much more optimal stability properties than the global reconstructions of [16], [17].

Therefore, Theorem 1.2 illustrating the progress in global stability of [17], [20] is much more optimal than Theorem 1.1 related with global reconstruction results of [16] and [17] only.

4. Faddeev functions

We consider the Faddeev functions G, ψ and h (see [7], [8], [10], [16]):

$$\psi(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G(x - y, k)v(y)\psi(y, k)dy, \tag{4.1}$$

$$G(x, k) = e^{ikx} g(x, k), \ \ g(x, k) = -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \tag{4.2}$$

where $x \in \mathbb{R}^d$, $k \in \Sigma$,

$$\Sigma = \{k \in \mathbb{C}^d : \quad k^2 = k_1^2 + \ldots + k_d^2 = 0\}; \tag{4.3}$$

$$h(k,l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx} v(x) \psi(x,k) dx, \tag{4.4}$$

where $(k,l) \in \Theta$,

$$\Theta = \{k \in \Sigma, \quad l \in \Sigma : \; Im\, k = Im\, l\}. \tag{4.5}$$

One can consider (4.1), (4.4) assuming that v is a sufficiently regular function on $\mathbb{R}^d$ with sufficient decay at infinity. For example, one can consider (4.1), (4.4) assuming that (1.2) holds.

We recall that:

$$\Delta G(x,k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \Sigma; \tag{4.6}$$

formula (4.1) at fixed k is considered as an equation for

$$\psi = e^{ikx} \mu(x,k), \tag{4.7}$$

where μ is sought in $L^\infty(\mathbb{R}^d)$; as a corollary of (4.1),(4.2), (4.6), ψ satisfies (1.1); h of (4.4) is a generalized "scattering" amplitude in the complex domain at zero energy.

Note that, actually, G, ψ, h of (4.1)-(4.5) are zero energy restrictions of functions introduced by Faddeev as extentions to the complex domain of some functions of the classical scattering theory for the Schrödinger equation at positive energies. In addition, G, ψ, h in their zero energy restriction were considered for the first time in [2]. The Faddeev functions G, ψ, h were, actually, rediscovered in [2].

We recall also that, under the assumptions of Theorem 2.1,

$$\mu(x,k) \to 1 \quad \text{as} \quad |Im\, k| \to \infty \quad (\text{uniformly in} \quad x) \tag{4.8}$$

and, for any $\sigma > 1$,

$$|\mu(x,k)| < \sigma \quad \text{for} \quad |Im\, k| \geq r_1(N,D,m,\sigma), \tag{4.9}$$

where $x \in \mathbb{R}^d$, $k \in \Sigma$;

$$\hat{v}(p) = \lim_{\substack{(k,l)\in\Theta,\; k-l=p \\ |Im\, k| = |Im\, l| \to \infty}} h(k,l) \quad \text{for any} \quad p \in \mathbb{R}^d, \tag{4.10}$$

$$\begin{aligned} &|\hat{v}(p) - h(k,l)| \leq \frac{c_3(D,m)N^2}{\rho} \quad \text{for} \quad (k,l) \in \Theta, \; p = k - l, \\ &|Im\, k| = |Im\, l| = \rho \geq r_2(N,D,m), \end{aligned} \tag{4.11}$$

where

$$\hat{v}(p) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \tag{4.12}$$

Results of the type (4.8), (4.9) go back to [2]. Results of the type (4.10), (4.11) (with less precise right-hand side in (4.11)) go back to [10]. Estimates (4.8), (4.11) are related also with some important L_2-estimate going back to [23] on the Green function g of (4.1).

Note also that in some considerations it is convenient to consider h on Θ as H on Ω, where

$$\begin{aligned} h(k,l) &= H(k,k-l), \quad (k,l)\in\Theta, \\ H(k,p) &= h(k,k-p), \quad (k,p)\in\Omega, \end{aligned} \tag{4.13}$$

$$\Omega = \{k\in\mathbb{C}^d,\ p\in\mathbb{R}^d:\ k^2=0,\ p^2=2kp\}. \tag{4.14}$$

For more information on properties of the Faddeev functions G, ψ, h, see [10], [17], [20] and references therein.

In the next section we recall that Problem 1.1 (of Introduction) admits a reduction to the following inverse "scattering" problem:

Problem 4.1. Given h on Θ, find v on $\mathbb{R}^d$.

5. Reduction of [16], [17] of Problem 1.1 to Problem 4.1

Let conditions (1.2), (1.4) hold for potentials v_1 and v_2, where D is fixed. Let Φ_i, ψ_i, h_i denote the DtN map Φ and the Faddeev functions ψ, h for $v=v_i$, $i=1,2$. Let also $\Phi_i(x,y)$ denote the Schwartz kernel $\Phi(x,y)$ of the integral operator Φ for $v=v_i$, $i=1,2$. Then (see [17] for details):

$$h_2(k,l)-h_1(k,l) = \big(\frac{1}{2\pi}\big)^d \int\limits_{\partial D}\int\limits_{\partial D} \psi_1(x,-l)(\Phi_2-\Phi_1)(x,y)\psi_2(y,k)dydx, \tag{5.1}$$

where $(k,l)\in\Theta$;

$$\psi_2(x,k) = \psi_1(x,k) + \int_{\partial D} A(x,y,k)\psi_2(y,k)dy,\ x\in\partial D, \tag{5.2a}$$

$$A(x,y,k) = \int_{\partial D} R_1(x,z,k)(\Phi_2-\Phi_1)(z,y)dz,\ x,y\in\partial D, \tag{5.2b}$$

$$R_1(x,y,k) = G(x-y,k) + \int_{\mathbb{R}^d} G(x-z,k)v_1(z)R_1(z,y,k)dz,\ x,y\in\mathbb{R}^d, \tag{5.3}$$

where $k\in\Sigma$. Note that: (5.1) is an explicit formula, (5.2a) is considered as an equation for finding ψ_2 on ∂D from ψ_1 on ∂D and A on $\partial D\times\partial D$ for each fixed k, (5.2b) is an explicit formula, (5.3) is an equation for finding R_1 from G and v_1, where G is the function of (4.2).

Note that formulas and equations (5.1)-(5.3) for $v_1\equiv 0$ were given in [16] (see also [10] (Note added in proof), [14], [15]). In this case $h_1\equiv 0$, $\psi_1 = e^{ikx}$, $R_1 = G(x-y,k)$. Formulas and equations (5.1)-(5.3) for the general case were given in [17].

Formulas and equations (5.1)-(5.3) with fixed background potential v_1 reduce Problem 1.1 (of Introduction) to Problem 4.1 (of Section 4). In this connection we consider (5.1)-(5.3) for $v_2=v$, $\psi_2=\psi$, $h_2=h$ and this reduction consists of the following steps:

(1) $v_1\to\Phi_1$, ψ_1, R_1, h_1 via formulas and equations of the diret problem for v_1;

(2) Φ, Φ_1, $R_1\big|_{\partial D\times\partial D\times\Sigma} \to A$ via (5.2b);
(3) A, $\psi_1\big|_{\partial D\times\Sigma} \to \psi\big|_{\partial D\times\Sigma}$ via (5.2a);
(4) h_1, $\psi_1\big|_{\partial D\times\Sigma}$, $\psi\big|_{\partial D\times\Sigma}$, Φ, $\Phi_1 \to h$ via (5.1).
Here, on the step (4) we find h on Θ for the unknown potential v of Problem 1.1.

6. Some considerations related with Θ and Ω

6.1 Some subsets of Θ *and* Ω. Let

$$\begin{aligned}&\mathcal{B}_r = \{p\in\mathbb{R}^d:\ |p|<r\},\quad \partial\mathcal{B}_r = \{p\in\mathbb{R}^d:\ |p|=r\},\\ &\bar{\mathcal{B}}_r = \mathcal{B}_r\cup\partial\mathcal{B}_r,\quad \text{where}\ \ r>0.\end{aligned}\tag{6.1}$$

In addition to Θ of (4.5), we consider, in particular, the following its subsets:

$$\begin{aligned}&\Theta_\rho = \{(k,l)\in\Theta:\ |Im\,k| = |Im\,l| < \rho\},\\ &b\Theta_\rho = \{(k,l)\in\Theta:\ |Im\,k| = |Im\,l| = \rho\},\\ &\bar{\Theta}_\rho = \Theta_\rho\cup b\Theta_\rho,\\ &\Theta^\infty_{\rho,\tau} = \{(k,l)\in\Theta\backslash\bar{\Theta}_\rho:\ k-l\in\mathcal{B}_{2\rho\tau}\},\\ &b\Theta_{\rho,\tau} = \{(k,l)\in b\Theta_\rho:\ k-l\in\mathcal{B}_{2\rho\tau}\},\end{aligned}\tag{6.2}$$

where $\rho>0$, $0<\tau<1$, and $\mathcal{B}_r$ is defined in (6.1).

In addition to Ω of (4.14), we consider, in particular, the following its subsets:

$$\begin{aligned}&\Omega_\rho = \{(k,p)\in\Omega:\ |Im\,k|<\rho\},\\ &b\Omega_\rho = \{(k,p)\in\Omega:\ |Im\,k|=\rho\},\\ &\bar{\Omega}_\rho = \Omega_\rho\cup b\Omega_\rho,\\ &\Omega^\infty_{\rho,\tau} = \{(k,p)\in\Omega\backslash\bar{\Omega}_\rho:\ p\in\mathcal{B}_{2\rho\tau}\},\\ &b\Omega_{\rho,\tau} = \{(k,p)\in b\Omega_\rho:\ p\in\mathcal{B}_{2\rho\tau}\},\end{aligned}\tag{6.3}$$

where $\rho>0$, $0<\tau<1$, and $\mathcal{B}_r$ is defined in (6.1).

Note that

$$\begin{aligned}&\Omega\approx\Theta,\ \Omega_\rho\approx\Theta_\rho,\ b\Omega_\rho\approx b\Theta_\rho,\\ &\Omega^\infty_{\rho,\tau}\approx\Theta^\infty_{\rho,\tau},\ b\Omega_{\rho,\tau}\approx b\Theta_{\rho,\tau},\end{aligned}\tag{6.4}$$

or, more precisely,

$$\begin{aligned}&(k,p)\in\Omega\Rightarrow(k,k-p)\in\Theta,\quad (k,l)\in\Theta\Rightarrow(k,k-l)\in\Omega\\ &\text{and the same for}\ \ \Omega_\rho,\ b\Omega_\rho,\ \Omega^\infty_{\rho,\tau},\ b\Omega_{\rho,\tau}\\ &\text{and}\ \ \Theta_\rho,\ b\Theta_\rho,\ \Theta^\infty_{\rho,\tau},\ b\Theta_{\rho,\tau},\ \ \text{respectively, in place of}\ \ \Omega\ \ \text{and}\ \ \Theta.\end{aligned}\tag{6.5}$$

We consider also, in particular,

$$\begin{aligned}&\Omega_\nu = \{(k,p)\in\Omega:\ p\notin\mathcal{L}_\nu\},\\ &\Omega^\infty_{\rho,\tau,\nu} = \Omega^\infty_{\rho,\tau}\cap\Omega_\nu,\ b\Omega_{\rho,\tau,\nu} = b\Omega_{\rho,\tau}\cap\Omega_\nu,\end{aligned}\tag{6.6}$$

where

$$\mathcal{L}_\nu = \{p \in \mathbb{R}^d : \ p = t\nu, \ t \in \mathbb{R}\}, \tag{6.7}$$

$\nu \in \mathbb{S}^{d-1}$, $\rho > 0$, $0 < \tau < 1$.

6.2. Coordinates on Ω for $d = 3$. In this subsection we assume that $d = 3$ in formulas (4.5), (4.14), (6.1)-(6.7).

For $p \in \mathbb{R}^3\backslash\mathcal{L}_\nu$ we consider $\theta(p)$ and $\omega(p)$ such that

$$\begin{aligned}
&\theta(p), \omega(p) \ \text{ smoothly depend on } \ p \in \mathbb{R}^3\backslash\mathcal{L}_\nu, \\
&\text{take values in } \ \mathbb{S}^2, \ \text{ and} \\
&\theta(p)p = 0, \ \omega(p)p = 0, \ \theta(p)\omega(p) = 0,
\end{aligned} \tag{6.8}$$

where $\mathcal{L}_\nu$ is defined by (6.7) (for $d = 3$).

Assumptions (6.8) imply that

$$\omega(p) = \frac{p \times \theta(p)}{|p|} \ \text{ for } \ p \in \mathbb{R}^3\backslash\mathcal{L}_\nu \tag{6.9a}$$

or

$$\omega(p) = -\frac{p \times \theta(p)}{|p|} \ \text{ for } \ p \in \mathbb{R}^3\backslash\mathcal{L}_\nu, \tag{6.9b}$$

where $\times$ denotes vector product.

To satisfy (6.8), (6.9a) we can take

$$\theta(p) = \frac{\nu \times p}{|\nu \times p|}, \ \omega(p) = \frac{p \times \theta(p)}{|p|}, \ p \in \mathbb{R}^3\backslash\mathcal{L}_\nu. \tag{6.10}$$

Let θ, ω satisfy (6.8). Then (according to [19]) the following formulas give a diffeomorphism between Ω_ν and $(\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash\mathcal{L}_\nu)$:

$$(k, p) \to (\lambda, p), \ \text{ where } \ \lambda = \lambda(k, p) = \frac{2k(\theta(p) + i\omega(p))}{i|p|}, \tag{6.11a}$$

$$\begin{aligned}
&(\lambda, p) \to (k, p), \ \text{ where } \ k = k(\lambda, p) = \kappa_1(\lambda, p)\theta(p) + \kappa_2(\lambda, p)\omega(p) + \frac{p}{2}, \\
&\kappa_1(\lambda, p) = \frac{i|p|}{4}(\lambda + \frac{1}{\lambda}), \ \ \kappa_2(\lambda, p) = \frac{|p|}{4}(\lambda - \frac{1}{\lambda}),
\end{aligned} \tag{6.11b}$$

where $(k, p) \in \Omega_\nu$, $(\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash\mathcal{L}_\nu)$. In addition, formulas (6.11a), (6.11b) for $\lambda(k)$ and $k(\lambda)$ at fixed $p \in \mathbb{R}^3\backslash\mathcal{L}_\nu$ give a diffeomorphism between $Z_p = \{k \in \mathbb{C}^3 : \ (k, p) \in \Omega\}$ for fixed p and $\mathbb{C}\backslash 0$.

In addition, for k and λ of (6.11) we have that

$$|Im\, k| = \frac{|p|}{4}\big(|\lambda| + \frac{1}{|\lambda|}\big), \ \ |Re\, k| = \frac{|p|}{4}\big(|\lambda| + \frac{1}{|\lambda|}\big), \tag{6.12}$$

where $(k,p)\in\Omega_\nu$, $(\lambda,p)\in(\mathbb{C}\backslash 0)\times(\mathbb{R}^3\backslash\mathcal{L}_\nu)$.

Let

$$\Lambda_{\rho,\nu}=\{(\lambda,p):\ \lambda\in\mathcal{D}_{\rho/|p|},\ \ p\in\mathbb{R}^3\backslash\mathcal{L}_\nu\},\tag{6.13}$$

$$\Lambda_{\rho,\tau,\nu}=\{(\lambda,p):\ \lambda\in\mathcal{D}_{\rho/|p|},\ \ p\in\mathbb{R}^3\backslash\mathcal{L}_\nu,\ \ |p|<2\tau\rho\},$$
$$b\Lambda_{\rho,\tau,\nu}=\{(\lambda,p):\ \lambda\in\mathcal{T}_{\rho/|p|},\ \ p\in\mathbb{R}^3\backslash\mathcal{L}_\nu,\ \ |p|<2\tau\rho\},$$

where $\rho>0$, $0<\tau<1$, $\nu\in\mathbb{S}^2$,

$$\mathcal{D}_r=\{\lambda\in\mathbb{C}\backslash 0:\ \frac{1}{4}(|\lambda|+|\lambda|^{-1})>r\},\ r>0,\tag{6.14}$$

$$\mathcal{T}_r=\{\lambda\in\mathbb{C}:\ \frac{1}{4}(|\lambda|+|\lambda|^{-1})=r\},\ r\geq 1/2,\tag{6.15}$$

$\mathcal{L}_\nu$ is defined by (6.7) (for $d=3$).

Note that

$$\begin{aligned}&\Lambda_{\rho,\tau,\nu}=\Lambda^+_{\rho,\tau,\nu}\cup\Lambda^-_{\rho,\tau,\nu},\ \Lambda^+_{\rho,\tau,\nu}\cap\Lambda^-_{\rho,\tau,\nu}=\emptyset,\\&b\Lambda_{\rho,\tau,\nu}=b\Lambda^+_{\rho,\tau,\nu}\cup b\Lambda^-_{\rho,\tau,\nu},\end{aligned}\tag{6.16}$$

where

$$\begin{aligned}&\Lambda^\pm_{\rho,\tau,\nu}=\{(\lambda,p):\ \ \lambda\in\mathcal{D}^\pm_{\rho/|p|},\ \ p\in\mathcal{B}_{2\tau\rho}\backslash\mathcal{L}_\nu\},\\&b\Lambda^\pm_{\rho,\tau,\nu}=\{(\lambda,p):\ \ \lambda\in\mathcal{T}^\pm_{\rho/|p|},\ \ p\in\mathcal{B}_{2\tau\rho}\backslash\mathcal{L}_\nu\},\end{aligned}\tag{6.17}$$

$$\begin{aligned}&\mathcal{D}^\pm_r=\{\lambda\in\mathbb{C}\backslash 0:\ \ \frac{1}{4}(|\lambda|+|\lambda|^{-1})>r,\ \ |\lambda|^{\pm1}<1\},\\&\mathcal{T}^\pm_r=\{\lambda\in\mathbb{C}:\ \ \frac{1}{4}(|\lambda|+|\lambda|^{-1})=r,\ \ |\lambda|^{\pm1}\leq 1\},\ r>1/2,\end{aligned}\tag{6.18}$$

where $\rho>0$, $\tau\in]0,1[$, $\nu\in\mathbb{S}^2$.

Using (6.12) one can see that formulas (6.11) give also the following diffeomorphisms

$$\begin{aligned}&\Omega_\nu\backslash\bar{\Omega}_\rho\approx\Lambda_{\rho,\nu},\ \ \Omega^\infty_{\rho,\tau,\nu}\approx\Lambda_{\rho,\tau,\nu},\\&b\Omega_{\rho,\tau,\nu}\approx b\Lambda_{\rho,\tau,\nu},\\&Z^\infty_{p,\rho}=\{k\in\mathbb{C}^3:\ (k,p)\in\Omega_\nu\backslash\bar{\Omega}_\rho\}\approx\mathcal{D}_{\rho/|p|}\ \text{ for fixed }\ p,\end{aligned}\tag{6.19}$$

where $\rho>0$, $0<\tau<1$, $\nu\in\mathbb{S}^2$ (and where we use the definitions (6.3), (6.6), (6.13)).

In [19] λ,p of (6.11) were used as coordinates on Ω. In the present work we use them also as coordinates on $\Omega\backslash\Omega_\rho$ (or more precisely on $\Omega_\nu\backslash\Omega_\rho$).

7. An integral equation of [20] and some related formulas

In the main considerations of [20] it is assumed that $d=3$ and the basic assumption on v consists in the following condition on its Fourier transform:

$$\hat{v}\in L^\infty_\mu(\mathbb{R}^3)\cap\mathcal{C}(\mathbb{R}^3)\ \text{ for some real }\ \mu\geq 2,\tag{7.1}$$

where $\hat{v}$ is defined by (4.12) (for $d = 3$),

$$L^\infty_\mu(\mathbb{R}^d) = \{u \in L^\infty(\mathbb{R}^d) : \ \ \|u\|_\mu < +\infty\},$$
$$\|u\|_\mu = ess \sup_{p\in\mathbb{R}^d} (1 + |p|)^\mu |u(p)|, \ \ \mu > 0, \tag{7.2}$$

and $\mathcal{C}$ denotes the space of continuous functions.

Note that

$$v \in W^{m,1}(\mathbb{R}^d) \Longrightarrow \hat{v} \in L^\infty_\mu(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d),$$
$$\|\hat{v}\|_\mu \le c_4(m, d)\|v\|_{m,1} \ \text{ for } \ \mu = m, \tag{7.3}$$

where $W^{m,1}$, L^∞_μ are the spaces of (2.2), (7.2).

Let

$$H(\lambda, p) = H(k(\lambda, p), p), \ \ (\lambda, p) \in (\mathbb{C}\backslash 0) \times (\mathbb{R}^3\backslash\mathcal{L}_\mu), \tag{7.4}$$

where H is the function of (4.13), λ, p are the coordinates of Subsection 6.2 under assumption (6.9a).

Let

$$L^\infty_\mu(\Lambda_{\rho,\tau,\nu}) = \{U \in L^\infty(\Lambda_{\rho,\tau,\nu}) : \ \ |||U|||_{\rho,\tau,\mu} < \infty\},$$
$$|||U|||_{\rho,\tau,\mu} = ess \sup_{(\lambda,p)\in\Lambda_{\rho,\tau,\nu}} (1 + |p|)^\mu |U(\lambda, p)|, \ \ \mu > 0, \tag{7.5}$$

where $\Lambda_{\rho,\tau,\nu}$ is defined in (6.13), $\rho > 0$, $\tau \in]0, 1[$, $\nu \in \mathbb{S}^2$, $\mu > 0$.

Let v satisfy (7.1) and $\|\hat{v}\|_\mu \le C$. Let

$$\eta(C, \rho, \mu) \stackrel{\text{def}}{=} a(\mu)C(\ln\rho)^2\rho^{-1} < 1, \ \ln\rho \ge 2, \tag{7.6}$$

where $a(\mu)$ is the constant $c_2(\mu)$ of [20]. Let $H(\lambda, p)$ be defined by (7.4) and be considered as a function on $\Lambda_{\rho,\tau,\nu}$ of (6.13). Then (see Section 5 of [20]):

$$H = H^0 + M_{\rho,\tau}(H) + Q_{\rho,\tau}, \ \ \tau \in]0, 1[, \tag{7.7}$$

where

$$H^0(\lambda, p) = \frac{1}{2\pi i} \int\limits_{\mathcal{T}^+_{\rho/|p|}} H(\zeta, p)\frac{d\zeta}{\zeta - \lambda}, \ (\lambda, p) \in \Lambda^+_{\rho,\tau,\nu}, \tag{7.8a}$$

$$H^0(\lambda, p) = -\frac{1}{2\pi i} \int\limits_{\mathcal{T}^-_{\rho/|p|}} H(\zeta, p)\frac{\lambda d\zeta}{\zeta(\zeta - \lambda)}, \ (\lambda, p) \in \Lambda^-_{\rho,\tau,\nu}, \tag{7.8b}$$

where $\Lambda^\pm_{\rho,\tau,\nu}$, $\mathcal{T}^\pm_r$ are defined in (6.17), (6.18) (and where the integrals along $\mathcal{T}^\pm_r$ are taken in the counter-clock wise direction);

$$M_{\rho,\tau}(U)(\lambda, p) = M^+_{\rho,\tau}(U)(\lambda, p) =$$
$$-\frac{1}{\pi} \int\int\limits_{\mathcal{D}^+_{\rho/|p|}} (U, U)_{\rho,\tau}(\zeta, p)\frac{dRe\,\zeta dIm\,\zeta}{\zeta - \lambda}, \ (\lambda, p) \in \Lambda^+_{\rho,\tau,\nu}, \tag{7.9a}$$

$$M_{\rho,\tau}(U)(\lambda,p) = M^{-}_{\rho,\tau}(U)(\lambda,p) =$$
$$-\frac{1}{\pi}\iint\limits_{\mathcal{D}^{-}_{\rho/|p|}} (U,U)_{\rho,\tau}(\zeta,p)\frac{\lambda dRe\,\zeta dIm\,\zeta}{\zeta(\zeta-\lambda)},\ (\lambda,p)\in\Lambda^{-}_{\rho,\tau,\nu}, \tag{7.9b}$$

$$\begin{aligned}&(U_1,U_2)_{\rho,\tau}(\zeta,p) = \{\chi_{2\tau\rho}U_1',\chi_{2\tau\rho}U_2'\}(\zeta,p),\ (\zeta,p)\in\Lambda_{\rho,\tau,\nu},\\ &\chi_{2\tau\rho}U_j'(k,p) = U_j(\lambda(k,p),p),\ (k,p)\in\Omega^{\infty}_{\rho,\tau,\nu},\\ &\chi_{2\tau\rho}U_j'(k,p)=0,\quad |p|\geq 2\tau\rho,\quad j=1,2,\end{aligned} \tag{7.10}$$

where U, U_1, U_2 are test functions on $\Lambda_{\rho,\tau,\nu}$, $\Omega^{\infty}_{\rho,\tau,\nu}$ is defined in (6.6), $\lambda(k,p)$ is defined in (6.11a), $\{\cdot,\cdot\}$ is defined by the formula

$$\begin{aligned}\{F_1,F_2\}(\lambda,p) = -\frac{\pi}{4}\int\limits_{-\pi}^{\pi}\Big(\frac{|p|}{2}\frac{|\lambda|^2-1}{\bar\lambda|\lambda|}(\cos\varphi-1)-\frac{|p|}{\bar\lambda}\sin\varphi\Big)\times\\ F_1(k(\lambda,p),-\xi(\lambda,p,\varphi))F_2(k(\lambda,p)+\xi(\lambda,p,\varphi),p+\xi(\lambda,p,\varphi))d\varphi,\end{aligned} \tag{7.11}$$

for $(\lambda,p)\in\Lambda_{\rho,\nu}$, where F_1, F_2 are test functions on $\Omega\backslash\bar\Omega_\rho$, $k(\lambda,p)$ is defined in (6.11b), $\Lambda_{\rho,\nu}$ is defined in (6.13),

$$\xi(\lambda,p,\varphi) = Re\,k(\lambda,p)(\cos\varphi-1)+k^{\perp}(\lambda,p)\sin\varphi, \tag{7.12}$$
$$k^{\perp}(\lambda,p) = \frac{Im\,k(\lambda,p)\times Re\,k(\lambda,p)}{|Im\,k(\lambda,p)|}, \tag{7.13}$$

where $\times$ in (7.13) denotes vector product;

$$H, H^0, Q_{\rho,\tau}\in L^{\infty}_{\mu}(\Lambda_{\rho,\tau,\nu}), \tag{7.14}$$
$$|||H|||_{\rho,\tau,\mu_0}\leq \frac{C}{1-\eta(C,\rho,\mu)}, \tag{7.15a}$$
$$|||H^0|||_{\rho,\tau,\mu_0}\leq \frac{C}{1-\eta(C,\rho,\mu)}\big(1+\frac{c_5(\mu_0)C}{1-\eta(C,\rho,\mu)}\big), \tag{7.15b}$$
$$|||Q_{\rho,\tau}|||_{\rho,\tau,\mu_0}\leq \frac{3c_5(\mu_0)C^2}{(1-\eta(C,\rho,\mu))^2(1+2\tau\rho)^{\mu-\mu_0}}, \tag{7.15c}$$

where $2\leq\mu_0\leq\mu$, c_5 is the constant b_4 of [20], $\eta(C,\rho,\mu)$ is defined by (7.6).

Following [20] we consider (7.7) as an approxiate integral equation for finding H on $\Lambda_{\rho,\tau,\nu}$ from H^0 on $\Lambda_{\rho,\tau,\nu}$ with unknown remainder $Q_{\rho,\tau}$.

Note also that if $\hat v$ satisfies (7.1), then (see [19], [20])

$$\begin{aligned}&H(\lambda,p)\to\hat v(p)\ \text{ as }\ \lambda\to 0,\\ &H(\lambda,p)\to\hat v(p)\ \text{ as }\ \lambda\to\infty,\end{aligned} \tag{7.16}$$

where $p\in\mathcal{B}_{2\tau\rho}\backslash\mathcal{L}_\nu$, H is defined by (7.4) and is considered as a function on $\Lambda_{\rho,\tau,\nu}$, $\rho>0$, $0<\tau<1$, $\nu\in\mathbb{S}^2$.

8. Proof of Theorem 2.2 for $d = 3$

Our proof of Theorem 2.2 for $d = 3$ is based on formulas (7.16) of Section 7 and on Lemma 8.4 given below in the present section. In turn, Lemma 8.4 follows from formula (7.3) of Section 7 and from Lemmas 8.1, 8.2, 8.3 given below in the present section. In turn:

(1) Lemma 8.1 follows from some estimates and Lemmas of [20] related with studies of the non-linear integral equation given as equation (7.7) of the present paper;
(2) Lemma 8.2 does not follow immediately from results of preceding works and its proof is given in Section 9;
(3) Lemma 8.3 follows from formulas (4.9), (5.1) and from definitions, see Section 9 for proof of this lemma.

Lemma 8.1. *Let $\hat{v}_i$ satisfy (7.1) and $\|\hat{v}_i\|_\mu < C$, where $i = 1, 2$. Let*

$$0 < \tau \le \tau_1(\mu, \mu_0, C, \delta), \;\; \rho \ge \rho_1(\mu, \mu_0, C, \delta), \tag{8.1}$$

where τ_1, ρ_1 are the constants of Section 4 of [20] and where $\delta = 1/2$, $2 \le \mu_0 < \mu$. Then

$$|||H_2 - H_1|||_{\rho,\tau,\mu_0} \le 2(|||H_2^0 - H_1^0|||_{\rho,\tau,\mu_0} + |||Q^2_{\rho,\tau} - Q^1_{\rho,\tau}|||_{\rho,\tau,\mu_0}), \tag{8.2}$$

where H_i, H_i^0, $Q^i_{\rho,\tau}$ are the functions of (7.4), (7.7), (7.8), (7.14), (7.15) for $v = v_i$, $i = 1, 2$, $|||\cdot|||_{\rho,\tau,\mu_0}$ is defined as in (7.5). In addition,

$$|||Q^2_{\rho,\tau} - Q^1_{\rho,\tau}|||_{\rho,\tau,\mu_0} \le \frac{24 c_5(\mu_0) C^2}{(1 + 2\tau\rho)^{\mu-\mu_0}}. \tag{8.3}$$

In connection with (8.1) we remind that $\tau_1 \in]0, 1[$ is sufficiently small and ρ_1 is sufficiently great, see [20].

Lemma 8.1 follows from estimates mentioned as estimates (7.14), (7.15) of the present paper (see estimates (4.3), (5.20), (5.22) of [20]) and from Lemmas A.1, A.2, A.3 and estimate (A.5) (see Appendix to the present paper).

Lemma 8.2. *Let $\hat{v}_i$ satisfy (7.1) and $\|\hat{v}_i\|_\mu < C$, where $i = 1, 2$. Let*

$$\eta(C, \rho_0, \mu) \le 1/2, \quad \ln \rho_0 \ge 2, \tag{8.4}$$

where η is defined in (7.6). Let

$$0 < \tau_0 < 1, \quad 2 \le \mu_0 < \mu, \quad \rho = 2\rho_0, \quad \tau = \tau_0/2.$$

Then

$$|||H_2^0 - H_1^0|||_{\rho,\tau,\mu_0} \le (c_6 + 4c_7(\mu_0, \tau_0, \rho_0)C)\Delta_{\rho_0,\rho,\mu_0}, \tag{8.5}$$

$$\Delta_{\rho_0,\rho,\mu_0} \stackrel{\mathrm{def}}{=} |||\chi_{\rho_0,\rho}(H_2 - H_1)|||_{\rho_0,\mu_0}, \tag{8.6}$$

where H_i, H_i^0 are the functions of (4.13), (7.8) for $v = v_i$, $i = 1, 2$, $\chi_{\rho_0,\rho}$ is the characteristic function of $\bar{\Omega}_\rho \backslash \bar{\Omega}_{\rho_0}$, $|||\cdot|||_{\rho,\tau,\mu_0}$ in (8.5) is defined as in (7.5), $|||\cdot|||_{\rho_0,\mu_0}$ in (8.6) is defined *as in* (A.12) of Appendix, c_6 *is defined by* (9.9), c_7 *is the constant* c_8 *of* [20] (*that is* $c_7(\mu,\tau,\rho) = 3b_1(\mu)\tau^2 + 4b_2(\mu)\rho^{-1} + 4b_3(\mu)\tau$, *where* b_1, b_2, b_3 *are the constants of* [20]).

Lemma 8.2 is proved in Section 9.

Lemma 8.3. *Let the assumptions of Theorem* 2.1 *hold (for* $d = 3$). *Let*

$$\rho_0 \geq r_1(N, D, m, \sigma) \ \ \textit{for some} \ \ \sigma > 1, \tag{8.7}$$

where r_1 *is the number of* (4.9). *Let* $0 < \tau_0 < 1$, $0 < \mu_0$, $\rho = 2\rho_0$, $\tau = \tau_0/2$. *Then*

$$\Delta_{\rho_0,\rho,\mu_0} \leq c_8\sigma^2 e^{2\rho L}\|\Phi_2 - \Phi_1\|(1 + 2\rho)^{\mu_0}, \tag{8.8}$$

where $\Delta_{\rho_0,\rho,\mu_0}$ *is defined by* (8.6),

$$c_8 = (2\pi)^{-d}\int\limits_{\partial D} dx, \quad L = \max_{x\in\partial D}|x|, \tag{8.9}$$

$\|\Phi_2 - \Phi_1\|$ *is defined according to* (2.4).

Lemma 8.3 is proved in Section 9.

Lemma 8.4. *Let the assumptions of Theorem* 2.1 *hold (for* $d = 3$). *Let*

$$0 < \tau \leq \tau_2(m, \mu_0, N), \quad \rho \geq \rho_2(m, \mu_0, N, D, \sigma), \tag{8.10}$$

where $2 \leq \mu_0 < m$, $\sigma > 1$, *and* τ_2, ρ_2 *are constants such that* (8.10) *implies that*

$$\tau \leq \tau_1(m, \mu_0, c_4(m,3)N, 1/2), \quad \rho \geq \rho_1(m, \mu_0, c_4(m,3)N, 1/2), \tag{8.11a}$$

$$\tau < 1/2, \ \eta(c_4(m,3)N, \rho/2, m) \leq 1/2, \ \ln(\rho/2) \geq 2, \tag{8.11b}$$

$$\rho/2 \geq r_1(N, D, m, \sigma), \tag{8.11c}$$

where τ_1, ρ_1, η, r_1 *are the same that in* (8.1), (8.4), (8.7), c_4 *is the constant of* (7.3). *Then*

$$|||H_2 - H_1|||_{\rho,\tau,\mu_0} \leq c_9(N, D, m, \mu_0, \sigma, \tau)e^{2L\rho}\rho^{\mu_0}\|\Phi_2 - \Phi_1\| + c_{10}(N, m, \mu_0, \tau)\rho^{-(m-\mu_0)}, \tag{8.12}$$

where c_9, c_{10} *are some constants which can be given explicitly.*

Lemma 8.4 follows from formula (7.3) and Lemmas 8.1, 8.2, 8.3.

The final part of the proof of Theorem 2.2 for $d = 3$ consists of the following. Using the inverse Fourier transform formula

$$v(x) = \int\limits_{\mathbb{R}^3} e^{-ipx}\hat{v}(p)dp, \quad x \in \mathbb{R}^3, \tag{8.13}$$

we have that

$$\begin{aligned} \|v_1 - v_2\|_{L^\infty(D)} &\leq \sup_{x\in\bar{D}}\Big|\int\limits_{\mathbb{R}^3} e^{-ipx}(\hat{v}_2(p) - \hat{v}_1(p))dp\Big| \leq \\ & I_1(r) + I_2(r) \ \ \text{for any} \ \ r > 0, \end{aligned} \tag{8.14}$$

where

$$I_1(r) = \int\limits_{|p|\le r} |\hat v_2(p) - \hat v_1(p)|dp,$$
$$I_2(r) = \int\limits_{|p|\ge r} |\hat v_2(p) - \hat v_1(p)|dp. \tag{8.15}$$

Using (7.3), (7.5), (7.14) for H, (7.16) we obtain that

$$|\hat v_2(p) - \hat v_1(p)| \le |||H_2 - H_1|||_{\rho,\tau,2}(1+|p|)^{-2} \tag{8.16}$$

for $|p| < 2\tau\rho$, at least, under the assumptions of Lemma 8.4 for $\mu_0 = 2$.

Using (7.3) we obtain that

$$|\hat v_2(p) - \hat v_1(p)| \le 2c_4(m,3)N(1+|p|)^{-m}, \quad p \in \mathbb{R}^3. \tag{8.17}$$

Using (8.14)-(8.17) we have that

$$\begin{aligned}
&\|v_1 - v_2\|_{L^\infty(D)} \le I_1(2\tau\rho) + I_2(2\tau\rho) \le \\
&|||H_2 - H_1|||_{\rho,\tau,2} \int\limits_{|p|\le 2\tau\rho} \frac{dp}{(1+|p|)^2} + 2c_4(m,3)N \int\limits_{|p|\ge 2\tau\rho} \frac{dp}{(1+|p|)^m} \le \\
&8\pi\tau\rho|||H_2 - H_1|||_{\rho,\tau,2} + \frac{8\pi c_4(m,3)N}{(m-3)(2\tau)^{m-3}}\frac{1}{\rho^{m-3}},
\end{aligned} \tag{8.18}$$

at least, under the assumptions of Lemma 8.4 for $\mu_0 = 2$.

Under the assumptions of Lemma 8.4 for $\mu_0 = 2$, using (8.18) and (8.12) we obtain that

$$\|v_1 - v_2\|_{L^\infty(D)} \le c_{11}(N,D,m,\sigma,\tau)e^{2L\rho}\rho^3\|\Phi_2 - \Phi_1\| + c_{12}(N,m,\tau)\rho^{-(m-3)}, \tag{8.19}$$

where c_{11}, c_{12} are related in a simple way with c_9, c_{10} for $\mu_0 = 2$.

Let now

$$\alpha \in]0,1[, \ \beta = \frac{1-\alpha}{2L}, \ \delta = \|\Phi_1 - \Phi_2\|, \ \rho = \beta\ln(3+\delta^{-1}), \tag{8.20}$$

where δ is so small that $\rho \ge \rho_2(m,2,N,D,\sigma)$, where ρ_2 is the constant of (8.10). Then, due to (8.19),

$$\begin{aligned}
&\|v_1 - v_2\|_{L^\infty(D)} \le c_{11}(N,D,m,\sigma,\tau)(3+\delta^{-1})^{2L\beta}(\beta\ln(3+\delta^{-1}))^3\delta + \\
&c_{12}(N,D,m,\tau)(\beta\ln(3+\delta^{-1}))^{-(m-3)} = \\
&c_{11}(N,D,m,\sigma,\tau)\beta^3(1+3\delta)^{1-\alpha}\delta^\alpha(\ln(3+\delta^{-1}))^3 + \\
&c_{12}(N,D,m,\tau)\beta^{-(m-3)}(\ln(3+\delta^{-1}))^{-(m-3)},
\end{aligned} \tag{8.21}$$

where σ, τ are the same that in (8.10) for $\mu_0 = 2$ and where α, β and δ are the same that in (8.20).

Using (8.21) we obtain that

$$\|v_1 - v_2\|_{L^\infty(D)} \le c_{13}(N, D, m)(\ln(3 + \|\Phi_1 - \Phi_2\|^{-1}))^{-(m-3)} \tag{8.22}$$

for $\delta = \|\Phi_1 - \Phi_2\| \le \delta_0(N, D, m)$, where δ_0 is sufficiently small positive constant. Estimate (8.22) in the general case (with modified c_{13}) follows from (8.22) for $\delta = \|\Phi_1 - \Phi_2\| \le \delta_0(N, D, m)$ and the property that $\|v_j\|_{L^\infty(D)} \le c_{14}(m)N$ (for $d = 3$).

Thus, Theorem 2.2 for $d = 3$ is proved.

9. Proofs of Lemmas 8.2 and 8.3

Proof of Lemma 8.2. Using the maximum principle for holomorphic functions it is sufficient to prove that

$$\begin{aligned}&\sup_{(\lambda,p)\in b\Lambda^{\pm}_{\rho,\tau,\nu}} (1+|p|)^{\mu_0}|H_2^0(\lambda(1\mp 0),p) - H_1^0(\lambda(1\mp 0),p)| \le \\ &(c_6 + 4c_7(\mu_0,\tau_0,\rho_0)C)\Delta_{\rho_0,\rho,\mu_0},\end{aligned} \tag{9.1}$$

where $b\Lambda^+_{\rho,\tau,\nu}$, $b\Lambda^-_{\rho,\tau,\nu}$ are defined in (6.17) (and where $H_i^0(\lambda(1-0),p)$, $i = 1, 2$, are considered for $(\lambda, p) \in b\Lambda^+_{\rho,\tau,\nu}$, $H_i^0(\lambda(1+0), p)$, $i = 1, 2$, are considered for $(\lambda, p) \in b\Lambda^-_{\rho,\tau,\nu}$).

Using (7.8) and the Sohotsky-Plemelj formula, we have that

$$H^0(\lambda(1-0),p) = \frac{1}{2\pi i}\int_{\mathcal{T}^+_{\rho/|p|}} H(\zeta,p)\frac{d\zeta}{\zeta-\lambda(1+0)} + H(\lambda,p),\ (\lambda,p)\in b\Lambda^+_{\rho,\tau,\nu}, \tag{9.2a}$$

$$H^0(\lambda(1+0),p) = -\frac{1}{2\pi i}\int_{\mathcal{T}^-_{\rho/|p|}} H(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda(1-0))} + H(\lambda,p),\ (\lambda,p)\in b\Lambda^-_{\rho,\tau,\nu} \tag{9.2b}$$

In addition, using the Cauchy-Green formula we have that

$$\begin{aligned}H(\lambda,p) = &-\frac{1}{2\pi i}\int_{\mathcal{T}^+_{\rho/|p|}} H(\zeta,p)\frac{d\zeta}{\zeta-\lambda(1+0)} + \frac{1}{2\pi i}\int_{\mathcal{T}^+_{\rho/|p|}} H(\zeta,p)\frac{d\zeta}{\zeta-\lambda} - \\ &\frac{1}{\pi}\int\int_{\mathcal{D}^+_{\rho_0/|p|}\setminus\mathcal{D}^+_{\rho/|p|}} \frac{\partial H(\zeta,p)}{\partial\bar\zeta}\frac{dRe\zeta dIm\zeta}{\zeta-\lambda},\ (\lambda,p)\in b\Lambda^+_{\rho,\tau,\nu},\end{aligned} \tag{9.3a}$$

$$\begin{aligned}H(\lambda,p) = &\frac{1}{2\pi i}\int_{\mathcal{T}^-_{\rho/|p|}} H(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda(1-0))} - \frac{1}{2\pi i}\int_{\mathcal{T}^-_{\rho/|p|}} H(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)} - \\ &\frac{1}{\pi}\int\int_{\mathcal{D}^-_{\rho_0/|p|}\setminus\mathcal{D}^-_{\rho/|p|}} \frac{\partial H(\zeta,p)}{\partial\bar\zeta}\frac{\lambda dRe\zeta dIm\zeta}{\zeta(\zeta-\lambda)},\ (\lambda,p)\in b\Lambda^-_{\rho,\tau,\nu}\end{aligned} \tag{9.3b}$$

(where the integrals along $\mathcal{T}_r^{\pm}$ are taken in the counter- clockwise direction). In addition (see equation (A.11) of Appendix),

$$\frac{\partial H(\zeta,p)}{\partial\bar\zeta} = \{H,H\}(\zeta,p),\ (\zeta,p)\in\Lambda_{\rho_0,\tau_0,\nu}, \tag{9.4}$$

where $H(\zeta,p)$ in the left-hand side of (9.4) is defined according to (7.4), H in the right-hand side of (9.4) is the function of (4.13), $\{\cdot,\cdot\}$ is defined by (7.11).

Using (9.2), (9.3) we obtain that

$$H^0(\lambda(1-0),p) = \frac{1}{2\pi i}\int\limits_{\mathcal{T}^+_{\rho_0/|p|}} H(\zeta,p)\frac{d\zeta}{\zeta-\lambda} - \tag{9.5a}$$

$$\frac{1}{\pi}\int\limits_{\mathcal{D}^+_{\rho_0/|p|}\setminus\mathcal{D}^+_{\rho/|p|}}\int \frac{\partial H(\zeta,p)}{\partial\bar\zeta}\frac{dRe\zeta dIm\zeta}{\zeta-\lambda},\ (\lambda,p)\in b\Lambda^+_{\rho,\tau,\nu},$$

$$H^0(\lambda(1+0),p) = -\frac{1}{2\pi i}\int\limits_{\mathcal{T}^-_{\rho_0/|p|}} H(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)} - \tag{9.5b}$$

$$\frac{1}{\pi}\int\limits_{\mathcal{D}^-_{\rho_0/|p|}\setminus\mathcal{D}^-_{\rho/|p|}}\int \frac{\partial H(\zeta,p)}{\partial\bar\zeta}\frac{\lambda dRe\zeta dIm\zeta}{\zeta(\zeta-\lambda)},\ (\lambda,p)\in b\Lambda^-_{\rho,\tau,\nu}.$$

Using (9.5), (9.4) for $H^0=H^0_i$, $H=H_i$, $i=1,2$, we obtain that:

$$(H^0_2-H^0_1)(\lambda(1-0),p) = A^+(\lambda,p)+B^+(\lambda,p), \tag{9.6a}$$

$$A^+(\lambda,p) = \frac{1}{2\pi i}\int\limits_{\mathcal{T}^+_{\rho_0/|p|}} (H_2-H_1)(\zeta,p)\frac{d\zeta}{\zeta-\lambda},$$

$$B^+(\lambda,p) = -\frac{1}{\pi}\int\limits_{\mathcal{D}^+_{\rho_0/|p|}\setminus\mathcal{D}^+_{\rho/|p|}}\int \big(\{H_2-H_1,H_2\}+\{H_1,H_2-H_1\}\big)(\zeta,p)\frac{dRe\zeta dIm\zeta}{\zeta-\lambda},$$

where $(\lambda,p)\in b\Lambda^+_{\rho,\tau,\nu}$;

$$(H^0_2-H^0_1)(\lambda(1+0),p) = A^-(\lambda,p)+B^-(\lambda,p), \tag{9.6b}$$

$$A^-(\lambda,p) = -\frac{1}{2\pi i}\int\limits_{\mathcal{T}^-_{\rho_0/|p|}} (H_2-H_1)(\zeta,p)\frac{\lambda d\zeta}{\zeta(\zeta-\lambda)},$$

$$B^-(\lambda,p) = -\frac{1}{\pi}\int\limits_{\mathcal{D}^-_{\rho_0/|p|}\setminus\mathcal{D}^-_{\rho/|p|}}\int \big(\{H_2-H_1,H_2\}+\{H_1,H_2-H_1\}\big)(\zeta,p)\frac{\lambda dRe\zeta dIm\zeta}{\zeta(\zeta-\lambda)},$$

where $(\lambda,p)\in b\Lambda^-_{\rho,\tau,\nu}$.

Estimates (9.1) follow from formulas (9.6) and from the estimates

$$|A^{\pm}(\lambda,p)|\le c_6(1+|p|)^{-\mu_0}\Delta_{\rho_0,\rho,\mu_0},\quad (\lambda,p)\in b\Lambda^{\pm}_{\rho,\tau,\nu}, \tag{9.7}$$
$$|B^{\pm}(\lambda,p)|\le 4c_7(\mu_0,\tau_0,\rho_0)C(1+|p|)^{-\mu_0}\Delta_{\rho_0,\rho,\mu_0},\quad (\lambda,p)\in b\Lambda^{\pm}_{\rho,\tau,\nu}, \tag{9.8}$$

where

$$\begin{aligned}&c_6=\sup_{r\in]1/2,+\infty[}\frac{q(r)}{q(r)-q(2r)},\\ &q(r)=2r\big(1-\big(1-\frac{1}{4r^2}\big)^{1/2}\big).\end{aligned} \tag{9.9}$$

Note that

$$0<c_6\le(2\sqrt{3}-3)^{-1}, \tag{9.10}$$

where c_6 is defined by (9.9). Estimate (9.10) follows from the formulas

$$c_6=\frac{1}{1-2\sigma},\quad \sigma=\sup_{\tau\in]0,1[}\frac{1-(1-(1/4)\tau)^{1/2}}{1-(1-\tau)^{1/2}}, \tag{9.11}$$

$$\begin{aligned}&(1-\tau)^{1/2}\le 1-(1/2)\tau,\ \ 1-(1/4)\tau\ge a(1-(1/4)\tau)+1-a,\\ &a=2(2-\sqrt{3}),\ \tau\in]0,1[.\end{aligned} \tag{9.12}$$

Estimates (9.7) follow from the property that

$$(\zeta,p)\in b\Lambda_{\rho_0,\tau_0,\nu}\ \text{ if }\ \zeta\in\mathcal{T}^{\pm}_{\rho_0/|p|},\ |p|<2\tau\rho,\ \rho=2\rho_0>0,\ \tau=\tau_0/2,\ 0<\tau_0<1, \tag{9.13}$$

the properties that

$$\begin{aligned}&H_i\in\mathcal{C}(\Lambda_{\rho_0,\tau_0,\nu}\cup b\Lambda_{\rho_0,\tau_0,\nu}),\ i=1,2,\\ &|(H_2-H_1)(\lambda,p)|\le(1+|p|)^{-\mu_0}\Delta_{\rho_0,\rho,\mu_0},\quad (\lambda,p)\in b\Lambda_{\rho_0,\tau_0,\nu}\end{aligned} \tag{9.14}$$

(see formulas (A.8), (A.9), (A.12), (6.19), (7.4)) and from the formulas

$$\frac{1}{2\pi}\int\limits_{\mathcal{T}^-_r}\frac{|\lambda||d\zeta|}{|\zeta||\zeta-\lambda|}=\frac{1}{2\pi}\int\limits_{\mathcal{T}^+_r}\frac{|d\zeta'|}{|\zeta'-\lambda'|}\le\frac{q(r)}{q(r)-q(2r)}, \tag{9.15}$$

$\lambda\in\mathcal{T}^-_{2r}$, $\lambda'\in\mathcal{T}^+_{2r}$, $r>1/2$. In turn, formulas (9.15) follow from the property that $z^{-1}\in\mathcal{T}^+_r$ if $z\in\mathcal{T}^-_r$ and from the formula that $q(r)$ is the radius of $\mathcal{T}^+_r$, where $r>1/2$.

Estimates (9.8) follow from formulas (A.8), (A.9) (for $\rho=\rho_0$), formula (8.4), Lemma A.5 (for $\rho=\rho_0$, $\mu=\mu_0$), Lemma A.6 (for $\rho=\rho_0$) and the property

$$\lambda\in\mathcal{T}^{\pm}_{\rho/|p|}\subset\mathcal{D}^{\pm}_{\rho_0/|p|}\ \text{ if }\ (\lambda,p)\in b\Lambda^{\pm}_{\rho,\tau,\nu},\ \rho=2\rho_0>0,\ \tau=\tau_0/2,\ 0<\tau_0<1. \tag{9.16}$$

Lemma 8.2 is proved.

Proof of Lemma 8.3. Using (A.12), the formulas

$$\Omega \approx \Theta, \;\; \Omega_\rho \approx \Theta_\rho, \; \bar{\Omega}_\rho \approx \bar{\Theta}_\rho$$

(see (6.4)), and formulas (6.2), (4.13), we have that

$$\Delta_{\rho_0,\rho,\mu_0} \leq \sup_{(k,l)\in\bar{\Theta}_\rho\backslash\Theta_{\rho_0}} (1+|k-l|)^{\mu_0}|h_2(k,l)-h_1(k,l)|, \tag{9.17}$$

where $\Delta_{\rho_0,\rho,\mu_0}$ is defined by (8.6), h_1, h_2 are the functions of Section 5.

Estimate (8.8) follows from formulas (9.17), (5.1), (4.7), (4.9), the property that $|k-l| \leq 2\rho$ for $(k,l) \in \bar{\Theta}_\rho$ and the property that $|e^{ikx}| \leq e^{\rho L}$, $|e^{ilx}| \leq e^{\rho L}$ for $k,l \in \bar{\Theta}_\rho$, $x \in \partial D$.

Lemma 8.3 is proved.

Appendix

In this appendix we recall some results of [20] used in Sections 8 and 9 of the present work.

Consider the equation

$$U = U^0 + M_{\rho,\tau}(U), \;\; \rho > 0, \; \tau \in]0,1[, \tag{A.1}$$

for unknown U, where U^0, U are functions on $\Lambda_{\rho,\tau,\nu}$ defined in (6.13), $M_{\rho,\tau}$ is the map defined by (7.9).

In particular, (7.7) can be written as (A.1) with $U = H$, $U^0 = H^0 + Q_{\rho,\tau}$.

Lemma A.1. *Let $\rho > 0$, $\nu \in \mathbb{S}^2$, $\tau \in]0,1[$, $\mu \geq 2$ and $0 < r < (2c_7(\mu,\tau,\rho))^{-1}$, where c_7 is the constant c_8 of [20] (that is $c_7(\mu,\tau,\rho) = 3b_1(\mu)\tau^2 + 4b_2(\mu)\rho^{-1} + 4b_3(\mu)\tau$, where b_1, b_2, b_3 are the constants of [20]). Let $U^0 \in L^\infty(\Lambda_{\rho,\tau,\nu})$ and $|||U^0|||_{\rho,\tau,\mu} \leq r/2$ (see definition (7.5)). Then equation (A.1) is uniquely solvable for $U \in L^\infty(\Lambda_{\rho,\tau,\nu})$, $|||U|||_{\rho,\tau,\mu} \leq r$, and U can be found by the method of successive approximations, in addition,*

$$|||U - (M_{\rho,\tau,U^0})^n(0)|||_{\rho,\tau,\mu} \leq \frac{r(2c_7(\mu,\tau,\rho)r)^n}{2(1-2c_7(\mu,\tau,\rho)r)}, \; n \in \mathbb{N}, \tag{A.2}$$

where M_{ρ,τ,U^0} denotes the map $U \to U^0 + M_{\rho,\tau}(U)$.

Lemma A.1 corresponds to Lemma 4.4 of [20].

Lemma A.2. *Let the assumptions of Lemma A.1 be fulfilled. Let also $\tilde{U}^0 \in L^\infty(\Lambda_{\rho,\tau,\nu})$, $|||\tilde{U}^0|||_{\rho,\tau,\mu} \leq r/2$, and $\tilde{U}$ denote the solution of (A.1) with U^0 replaced by $\tilde{U}^0$, where $\tilde{U} \in L^\infty(\Lambda_{\rho,\tau,\nu})$, $|||\tilde{U}|||_{\rho,\tau,\mu} \leq r$. Then*

$$|||U - \tilde{U}|||_{\rho,\tau,\mu} \leq (1-2c_7(\mu,\tau,\rho)r)^{-1}|||U^0 - \tilde{U}^0|||_{\rho,\tau,\mu}. \tag{A.3}$$

Lemma A.2 corresponds to Lemma 4.5 of [20].

Let

$$r_{min} \stackrel{\text{def}}{=} \frac{2C}{1-\eta(C,\rho,\mu)} + \frac{2b_4(\mu_0)C^2}{(1-\eta(C,\rho,\mu))^2}\left(1+\frac{3}{(1+2\tau\rho)^{\mu-\mu_0}}\right), \qquad (A.4)$$

where $C>0$, $\rho\geq e^2$, $2\leq\mu_0<\mu$, $\tau\in]0,1[$, η is defined as in (7.6), b_4 is the constant of [20] (that is $b_4(\mu)=\pi^{-1}(b_1(\mu)n_1+b_2(\mu)n_2+b_3(\mu)n_3)$, where b_1, b_2, b_3 are the constants of [20] and n_1, n_2, n_3 are the constants of Lemma 11.1 of [19]). In addition, from estimates (7.14), (7.15) of the present paper (see estimates (3.3), (4.20), (4.22) of [20]) it follows that

$$\begin{aligned} &|||H|||_{\rho,\tau,\mu_0} \leq r_{min} \\ &|||H^0|||_{\rho,\tau,\mu_0} + |||Q_{\rho,\tau}|||_{\rho,\tau,\mu_0} \leq r_{min}/2. \end{aligned} \qquad (A.5)$$

Lemma A.3. *Let $C>0$, $2\leq\mu_0<\mu$, $0<\delta<1$,*

$$0<\tau\leq\tau_1(\mu,\mu_0,C,\delta), \quad \rho\geq\rho_1(\mu,\mu_0,C,\delta), \qquad (A.6)$$

where τ_1, ρ_1 are the constants of (4.36) of [20] (in particular, $\tau_1\in]0,1[$ is sufficiently small and ρ_1 is sufficiently great). Then

$$\begin{aligned} &0\leq\eta(C,\rho,\mu)<\delta, \\ &0\leq 2c_7(\mu_0,\tau,\rho)r_{min}(\mu,\mu_0,\tau,\rho,C)<\delta. \end{aligned} \qquad (A.7)$$

Lemma A.3 corresponds to estimates (4.36) of [20].

Lemma A.4. *Let v satisfy (7.1), $\|\hat{v}\|_\mu\leq C$, ρ satisfy (7.6). Let H be the function of (4.13). Then:*

$$H\in\mathcal{C}(\Omega\backslash\Omega_\rho), \qquad (A.8)$$

$$|H(k,p)|\leq\frac{C}{(1-\eta(C,\rho,\mu))(1+|p|)^\mu}, \quad (k,p)\in\Omega\backslash\Omega_\rho, \qquad (A.9)$$

$$\hat{v}(p)=\lim_{\substack{|k|\to\infty,\\ (k,p)\in\Omega}} H(k,p), \ p\in\mathbb{R}^3, \qquad (A.10)$$

where $\mathcal{C}$ denotes the space of continuous functions, Ω, Ω_ρ are defined in (4.14), (6.3), η is defined as in (7.6), $|k|=((Re\,k)^2+(Im\,k)^2)^{1/2}$;

$$\frac{\partial}{\partial\bar{\lambda}}H(k(\lambda,p),p)=\{H,H\}(\lambda,p), \quad (\lambda,p)\in\Lambda_{\rho,\nu}, \qquad (A.11)$$

where $k(\lambda,p)$ is defined in (6.11b), $\Lambda_{\rho,\nu}$ is defined in (6.13), $\{\cdot,\cdot\}$ is defined by (7.11).

Lemma A.4 corresponds to formulas (3.2)-(3.4), (3.22) of [20]. Let

$$\begin{aligned} &L^\infty_\mu(\Omega\backslash\bar{\Omega}_\rho)=\{U\in L^\infty(\Omega\backslash\bar{\Omega}_\rho):\ |||U|||_{\rho,\mu}<+\infty\}, \\ &|||U|||_{\rho,\mu}=ess\sup_{(k,p)\in\Omega\backslash\bar{\Omega}_\rho}(1+|p|)^\mu|U(k,p)|, \ \rho>0, \ \mu>0, \end{aligned} \qquad (A.12)$$

where Ω, $\bar\Omega_\rho$ are defined in (4.14), (6.3).

Lemma A.5. *Let $U_1, U_2 \in L^\infty_\mu(\Omega\backslash\bar\Omega_\rho)$, where $\rho > 0$, $\mu \geq 2$. Let $\{\cdot,\cdot\}$ be defined by (7.11). Then*

$$\{U_1, U_2\} \in L^\infty_{local}(\Lambda_{\rho,\nu}), \tag{A.13}$$

$$\begin{gathered}|\{U_1,U_2\}(\lambda,p)| \leq \frac{|||U_1|||_{\rho,\mu}|||U_2|||_{\rho,\mu}}{(1+|p|)^\mu} b(\mu,|\lambda|,|p|)\\ for\ almost\ all\ \ (\lambda,p)\in\Lambda_{\rho,\nu},\ b(\mu,|\lambda|,|p|) = \\ \left(\frac{b_1(\mu)|\lambda|}{(|\lambda|^2+1)^2} + \frac{b_2(\mu)|p|||\lambda|^2-1|}{|\lambda|^2(1+|p|(|\lambda|+|\lambda|^{-1}))^2} + \frac{b_3(\mu)|p|}{|\lambda|(1+|p|(|\lambda|+|\lambda|^{-1}))}\right),\end{gathered} \tag{A.14}$$

where $\Lambda_{\rho,\nu}$ is defined in (6.13), b_1, b_2, b_3 are the constants of [20].

Lemma A.5 corresponds to estimates (3.26), (3.27) of [20].

Consider

$$\begin{aligned}&u_1(\zeta,s) = \frac{|\zeta|}{(|\zeta|^2+1)^2},\quad u_2(\zeta,s) = \frac{(|\zeta|^2+1)s}{|\zeta|^2(1+s(|\zeta|+|\zeta|^{-1}))^2},\\ &u_3(\zeta,s) = \frac{s}{|\zeta|(1+s(|\zeta|+|\zeta|^{-1}))},\end{aligned} \tag{A.15}$$

where $\zeta \in \mathbb{C}$, $s > 0$.

Note that

$$0 < b(\mu,|\lambda|,|p|) \leq \sum_{j=1}^{3} b_j(\mu) u_j(\lambda,|p|), \tag{A.16}$$

where b is the function of (A.14), $(\lambda,p) \in \Lambda_{\rho,\nu}$, $\rho > 0$, $\mu \geq 2$.

Lemma A.6. *Let u_1, u_2, u_3 be defined by (A.15). Then:*

$$\int\limits_{\mathcal{D}^-_r} u_j(\zeta,s)\frac{|\lambda|}{|\zeta|}\frac{dRe\,\zeta dIm\,\zeta}{|\zeta-\lambda|} = \int\limits_{\mathcal{D}^+_r} u_j(\zeta,s)\frac{dRe\,\zeta dIm\,\zeta}{|\zeta-\lambda^{-1}|}, \tag{A.17}$$

$j = 1,2,3$, $r > 1/2$, $\lambda \in \mathcal{D}^-_r$, $s > 0$,

$$\int\limits_{\mathcal{D}^+_{\rho/|p|}} u_j(\zeta,|p|)\frac{dRe\,\zeta dIm\,\zeta}{|\zeta-\lambda|} \leq \begin{cases} (3/4)\pi(|p|/\rho)^2 & for\ j=1,\\ 4\pi/\rho & for\ j=2,\\ 2\pi|p|/\rho & for\ j=3,\end{cases} \tag{A.18}$$

$0 < |p| < 2\tau\rho$, $0 < \tau < 1$, $\lambda \in \mathcal{D}^+_{\rho/|p|}$, *where $\mathcal{D}^\pm_r$ are defined in (6.18).*

Lemma A.6 corresponds to formulas (7.2), (7.10) of [20].

References

[1] G.Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal. **27** (1988), 153-172.

[2] R.Beals and R.R.Coifman, *Multidimensional inverse scattering and nonlinear partial differential equations*, Proc. Symp. Pure Math. **43** (1985), 45-70.

[3] R.M.Brown and G.Uhlmann, *Uniqueness in the inverse conductivity problem for non-smooth conductivities in two dimensions*, Comm. Partial Diff. Eq. **22** (1997), 1009-1027.

[4] A.L.Bukhgeim, *Recovering a potential from Cauchy data in the two-dimensional case*, J.Inverse Ill-Posed Probl. **16**(1) (2008), 19-33.

[5] A.-P.Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), pp.65-73, Soc. Brasil. Mat. Rio de Janeiro, 1980.

[6] L.D.Druskin, *The unique solution of the inverse problem in electrical surveying and electrical well logging for piecewise-constant conductivity*, Physics of the Solid Earth **18**(1) (1982), 51-53.

[7] L.D.Faddeev, *Growing solutions of the Schrödinger equation*, Dokl. Akad. Nauk SSSR **165** (1965), 514-517 (in Russian); English Transl.: Sov. Phys. Dokl. **10** (1966), 1033-1035.

[8] L.D.Faddeev, *Inverse problem of quantum scattering theory II*, Itogi Nauki i Tekhniki, Sovr. Prob. Math. **3** (1974), 93-180 (in Russian); English Transl.: J.Sov. Math. **5** (1976), 334-396.

[9] I.M.Gelfand, *Some problems of functional analysis and algebra*, Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, pp.253-276.

[10] G.M.Henkin and R.G.Novikov, *The $\bar{\partial}$- equation in the multidimensional inverse scattering problem*, Uspekhi Mat. Nauk **42(3)** (1987), 93-152 (in Russian); English Transl.: Russ. Math. Surv. **42(3)** (1987), 109-180.

[11] R.Kohn and M.Vogelius, *Determining conductivity by boundary measurements II, Interior results*, Comm. Pure Appl. Math. **38** (1985), 643-667.

[12] M.Lassas, J.L.Mueller and S.Siltanen, *Mapping properties of the nonlinear Fourier transform in dimension two*, Comm.Partial Differential Equations **32**(4-6) (2007), 591-610.

[13] N.Mandache, *Exponential instability in an inverse problem for the Schrödinger equation*, Inverse Problems **17** (2001), 1435-1444.

[14] A.I.Nachman, *Reconstructions from boundary measurements*, Ann. Math. **128** (1988), 531-576.

[15] A.I.Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann, Math. **142** (1995), 71-96.

[16] R.G.Novikov, *Multidimensional inverse spectral problem for the equation* $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, Funkt. Anal. i Pril. **22(4)** (1988), 11-22 (in Russian); English Transl.: Funct. Anal. and Appl. **22** (1988), 263-272.

[17] R.G.Novikov, *Formulae and equations for finding scattering data from the Dirichlet-to-Neumann map with nonzero background potential*, Inverse Problems **21** (2005), 257-270.

[18] R.G.Novikov, *The $\bar{\partial}$- approach to approximate inverse scattering at fixed energy in three dimensions*, International Mathematics Research Papers, **2005:6**, (2005), 287-349.

[19] R.G.Novikov, *On non-overdetermined inverse scattering at zero energy in three dimensions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **5** (2006), 279-328

[20] R.G.Novikov, *An effectivization of the global reconstruction in the Gel'fand-Calderon inverse problem in three dimensions*, Contemporary Mathematics, **494** (2009), 161-184.

[21] R.G.Novikov and N.N.Novikova, *On stable determination of potential by boundary measurements*, ESAIM: Proceedings **26** (2009), 94-99.

[22] V.P.Palamodov, *Gabor analysis of the continuum model for impedance tomography*, Ark.Mat. **40** (2002), 169-187.

[23] J.Sylvester and G.Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. Math. **125** (1987), 153-169.

[24] R.Novikov and M.Santacesaria, *A global stability estimate for the Gel'fand-Calderon inverse problem in two dimensions*, in preparation.